

Mathematics and Computer Science: Boolean Algebra

w05-Lec

Assembled for 204111
by Areerat Trongratsameethong

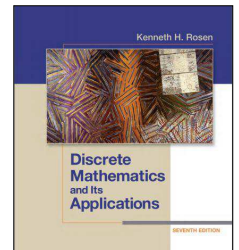
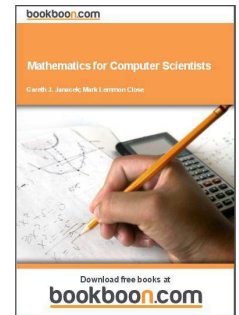
Introduction to Logical Value

- A statement has its logical value, either true (T) or false (F)
 - “it is raining”: may be **true** or **false**
 - “it is sunny”: may be **true** or **false**
- Many statements can be combined with **and** and **or** to be a compound statement.
 - “it is raining **and** it is sunny”
 - “it is raining **or** it is sunny”

The logical values of the compound statements depend on the logical value of each combined statement and what connective (“and” or “or”) is used.

Boolean Algebra

- **Mathematics for Computer Scientists – Janacek and Close**
 - Introduction to Logical Value
 - Logical Operators
- **Discrete Mathematics and Its Applications – K.H. Rosen**
 - Introduction to Boolean Algebra
 - Rules of Precedence for Boolean Operators
 - Boolean Properties
 - Boolean Expression Simplification



Logical Operators

- **Three logical operators are used in Boolean Algebra**
 - **Negation (not)**
 - **Conjunction (and)**
 - **Disjunction (or)**

Logical Operators (2)

- **Symbolic Notation:** is used to make things shorter
 - **Negation** denoted by \neg
 - **“and”** denoted by \wedge
 - **“or”** denoted by \vee
- A symbolic can be also used for a statement
 - p can be used for “it is raining”

5

Logical Operators (4)

- The logical values can be represented in form of truth table as an example below.

Example

Let p = “All computer scientists are men”

Two possible logical values of p are T and F

p	$\neg p$
T	F
F	T

Table 2.1: Truth table for negation (\neg)

7

Logical Operators (3)

- **Negation \neg**
 - The negation of a statement is false when the statement is true.
 - The negation of a statement is true when the statement is false.
- **Example**
 - Let p = “It is raining”, then $\neg p$ is “it is not raining”
 - If the logical value of p is F then logical value of $\neg p$ is T

6

Logical Operators (5)

- **Conjunction \wedge**
- If p and q are statements, then $p \wedge q$ is read as “ p and q ”.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 2.2: Truth table for \wedge

- Let
 - p = “It is green”,
 - q = “It is an apple” then
- $p \wedge q$ = “It is green and It is an apple”
- The logical value of $p \wedge q$ depends on each logical value of p and q as shown in Table 2.2.

8

Logical Operators (3)

- **Disjunction \vee**

- If p and q are statements, then

$p \vee q$ is read as " p or q ".

- Let

- p = "It is green",
- q = "It is an apple" then

$p \vee q$ = "It is green or It is an apple"

- The logical value of $p \vee q$ depends on each logical value of p and q as shown in Table 2.3.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Table 2.3: Truth table for \vee

9

Introduction to Boolean Algebra (2)

- The mathematical system written by Boole became known as Boolean algebra.
- All Boolean quantities have two possible outcomes: 1 or 0.
- There is no such thing as "2" or "-1" or "1/2" in the Boolean world.

11

Introduction to Boolean Algebra

- A Symbolic form of Aristotle's system of logic sought by George Boole (1815-1864) - The English mathematician
- Mathematical language dealing with the questions of logic
- An Investigation of the Laws of Thought (Boole 1854),
 - Theories of Logic and Probabilities
 - Mathematical Relationship Quantities Rule
 - true or false
 - 1 or 0

10

Introduction to Boolean Algebra (3)

- Boolean algebra as on-and-off circuits Control
- All signals are characterized as either "high" (1) or "low" (0).
- A Symbolic Analysis of Relay and Switching Circuits –MIT Thesis (Shannon 1938)
 - Mathematical tool for designing and analyzing digital circuits.
 - Defined the circuits in all electronic devices as 1 or 0 referring 'on' or in 'off' position.

12

Introduction to Boolean Algebra (4)

- Boolean algebra provides the operations and the rules for working with the set $\{0,1\}$.
- Operation for a circuit is called Boolean Function.
- Boolean Function produce output for each set of inputs.
- This function is built using Boolean expressions and operations.

13

Rules of Precedence for Boolean Operators (2)

Example: Find the value of $1 \wedge 0 \vee \sim (0 \vee 1)$

$$1 \wedge 0 \vee \sim (0 \vee 1)$$

$$1 \wedge 0 \vee \sim 1$$

$$1 \wedge 0 \vee 0$$

$$0 \vee 0$$

$$0$$

15

Rules of Precedence for Boolean Operators

- Order of Boolean Operators
 1. Complement (or Negation \neg) denoted by \sim
 2. Boolean Product denoted by \wedge
 3. Boolean Sum denoted by \vee

Note: Unless parentheses () are used, operations in the parentheses are done first.

14

Boolean Properties

- The simpler that we can make a Boolean function, the smaller the circuit that will result.
- Simpler Circuits
 - Cheaper to build
 - Consume less power
 - Run faster than the complex circuits
- With this in mind, we always want to reduce our Boolean functions to their **simplest form**.
- There are a number of Boolean identities that help us to do this.

16

Rules of Boolean Algebra

- Most Boolean identities have an AND (product) form as well as an OR (sum) form.
- We give our identities using both forms.

Identity Name	AND	OR
Identity Law	$1 \wedge x = x$	$0 \vee x = x$
Null Law	$0 \wedge x = 0$	$1 \vee x = 1$
Idempotent Law	$x \wedge x = x$	$x \vee x = x$
Inverse Law	$x \wedge \sim x = 0$	$x \vee \sim x = 1$

x	1	$1 \wedge x$
0	1	0
1	1	1

Truth table of $1 \wedge x$

How about the others?

Note: These laws can be proved by truth table.

17

Boolean Properties (3)

Identity Name	AND	OR
Absorption Law	$x \wedge (x \vee y) = x$	$x \vee (x \wedge y) = x$
Demorgan's Law	$\sim(x \wedge y) = \sim x \vee \sim y$	$\sim(x \vee y) = \sim x \wedge \sim y$
Double Complement Law	$\sim(\sim x) = x$	

Proof – Absorption Law:

$$x \wedge (x \vee y) = x$$

Rewrite: $\wedge \rightarrow \cdot, \vee \rightarrow +$

$$\begin{aligned} x \cdot (x + y) &= \underline{x \cdot x} + (x \cdot y) \\ &= x + (x \cdot y) \\ &= (x \cdot 1) + (x \cdot y) \\ &= x \cdot (1 + y) \\ &= x \cdot 1 \\ &= x \end{aligned}$$

Proof – Absorption Law:

$$x \vee (x \wedge y) = x$$

Rewrite: $\wedge \rightarrow \cdot, \vee \rightarrow +$

$$\begin{aligned} x + (x \cdot y) &= (x \cdot 1) + (x \cdot y) \\ &= x \cdot (1 + y) \\ &= x \cdot 1 \\ &= x \end{aligned}$$

19

Rules of Boolean Algebra(2)

Identity Name	AND
Commutative Law	$x \wedge y = y \wedge x$
Associative Law	$(x \wedge y) \wedge z = x \wedge (y \wedge z)$
Distributive Law	$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

Identity Name	OR
Commutative Law	$x \vee y = y \vee x$
Associative Law	$(x \vee y) \vee z = x \vee (y \vee z)$
Distributive Law	$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Remark: The above identities can be translated to logical equivalences about propositions and to identities about sets.

18

Boolean Expression Simplification

Example 1: $(x \vee y) \wedge (x \vee \sim y) \wedge \underline{\sim(x \wedge \sim z)}$

$$(x \vee y) \wedge (x \vee \sim y) \wedge \underline{\sim(x \wedge \sim z)}$$

DeMorgan's Law

$$\underline{(x \vee y) \wedge (x \vee \sim y)} \wedge \underline{\sim(x \wedge \sim z)}$$

Distributive Law

$$\underline{(x \wedge x) \vee (x \wedge \sim y) \vee (y \wedge x) \vee (y \wedge \sim y)} \wedge \underline{\sim(x \wedge \sim z)}$$

Idempotent and Inverse Laws

$$\underline{x \vee (x \wedge \sim y) \vee (y \wedge x) \vee 0} \wedge \underline{\sim(x \wedge \sim z)}$$

Absorption and Identity Laws

$$\underline{x \vee (y \wedge x)} \wedge \underline{\sim(x \wedge \sim z)}$$

Absorption Law

$$\underline{x \wedge (\sim x \vee z)}$$

Distributive Law

$$\underline{(x \wedge \sim x) \vee (x \wedge z)}$$

Inverse Law

$$\underline{0 \vee (x \wedge z)}$$

Identity Law

$$x \wedge z$$

20

Boolean Expression Simplification (2)

- **Example 2: Find the complement of the Boolean expression below. (Demorgan's Law)**

$$(x \wedge y) \vee (\sim x \wedge z) \vee (y \wedge \sim z)$$

$$\sim((x \wedge y) \vee (\sim x \wedge z) \vee (y \wedge \sim z))$$

$$\sim(x \wedge y) \wedge \sim(\sim x \wedge z) \wedge \sim(y \wedge \sim z)$$

$$(\sim x \vee \sim y) \wedge (x \vee \sim z) \wedge (\sim y \vee z)$$

21

Reference

- **Mathematics for Computer Scientists, Janacek and Close**
- **Discrete Mathematics and Its Applications, K.H. Rosen**

23

Summary

- **Introduction to Logical Value**
- **Logical Operators**
- **Introduction to Boolean Algebra**
- **Rules of Precedence for Boolean Operators**
- **Boolean Properties**
- **Boolean Expression Simplification**

22